

Understanding Partial Differential Equations

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Abstract — *In this paper, we present "a posteriori" analysis of the fundamental concepts involved in the modelling of problems of mathematical physics by Partial Differential Equations (PDEs). Our aim is to improve our students' understanding of PDEs when applied to an engineering problem, from a completely qualitative point of view. They should be able to understand the deep meaning of any Laplacian, curl, divergence, or gradient operator, as well as other differential terms, when appearing in any particular equation, besides its usefulness to model reality.*

Index Terms — *Partial differential equation, modelling, Laplacian, divergence, curl.*

INTRODUCTION

In this work, we try to improve the understanding of the basic concepts of PDEs from a qualitative point of view. Concepts like Laplacian operator, divergence, curl, and the time derivatives involved in diffusion, harmonics, wave equations and others are treated from an intuitive and graphical point of view. Always looking for a geometrical interpretation of the mathematical model problem. This attempt has been successfully reached in [1], [11] and [16]. Nevertheless, we think the idea still remains unapplied in undergraduate and graduate courses. So, this work tries to emphasize some graphical and intuitive concepts in order to become practical in lectures.

THE LAPLACIAN OPERATOR

It is well known the analogy between the slope of a curve given by its derivative at a point, and the gradient vector, which would be a generalization of this one-dimensional concept. Similarly, there is a relation between the convexity of a curve, given by its second derivative, and the convexity of a surface, given by its Laplacian. We may summarize this concept by stating that the Laplacian of a real function at a point is a measure of its convexity (or concavity), given by the difference between the value of the function at the point, and the average value of the function at neighbor points. More precisely, the Laplacian is negative when the value of the function is greater than the average of its neighborhood, indicating convexity (or a maximum, when conditions are fulfilled). It is noticeable, the analogy with convexity in one-dimensional problems, where a negative second derivative is an indicative of a downward curvature at that point. Similarly, a positive value of the Laplacian implies the existence of a concave form (or a minimum, if it is the case). The details of these assertions are derived from the fundamental definitions of derivatives and partial differentiation, and can be easily shown. We refer the interested reader to [16], [2], and [12].

"One immediate consequence of this definition is that a scalar function can have no maxima or minima in a region where the Laplacian vanishes. This is a result of considerable importance".

Equation (1), called Laplace's equation, "occurs in so many parts of physical theory that it is well to have a clear picture of its meaning".

Laplace's Equation

The case of a null Laplacian is much richer than the one-dimensional case of a null second derivative. In a real function of several variables, the null value of the Laplacian at a point would mean that there would be neither convexity nor concavity, as is the case of a saddle point, but with a lot of different spatial variations.

Let us consider, for example, the case of a very thin tent. At the point where the post holds up the tent, there is a maximum indicated by a negative Laplacian of the position of the cloth. However, at the rest of the tent, in our ideal case, the Laplacian is null, thus meaning that the tent cannot adopt any completely convex or concave form. If we observe a thin tent from a side, it appears to be concave in shape, like a slide, but if we take a cross section of it, then it appears clearly convex.

"The equation simply corresponds to the statement that the tension straightens out all the bulges in the cloth of the tent, so that the displacement at any point equals the average value of the displacement for neighboring points". One can say that the two-dimensional Laplacian operator measures the bulginess of the shape of the function".

"The generalization of this discussion to three dimensions is somewhat harder to picture but no more complicated in principle. We might picture the scalar function as corresponding to the concentration of a solute in a solvent. The three-dimensional analogue of bulginess might be termed lumpiness; if there is a tendency for the solute to lump together at any point, the Laplacian of the concentration will be negative there. In a case where the Laplacian vanishes, the solute has no lumpiness at all, its density arranging itself so as to average out as smoothly as possible the differences imposed by the boundary conditions. As in the two-dimensional case, the Laplace equation corresponds to the requirement that the value of the function at every point be equal to the average value of the function at neighboring points".

Poisson's Equation

The presence of electric charge density causes a concentration of the electric potential, so that the Laplacian of the potential is negative, and proportional to the charge density.

"The presence of a distributed source of heat in a solid causes a concentration of temperature, so that the Laplacian of the temperature is negative at that point, and proportional to the source of heat".

In a great many cases the scalar field is affected by a source function (which is itself another scalar field, obeying some other equations) according to (2). This equation is called Poisson's equation.

The Wave Equation

Equation (3) is called the wave equation for reasons which will shortly become apparent. It states that the transverse acceleration of any part of the medium (for example, a string) is proportional to the curvature of that part. "A wave may be roughly described as a configuration of the medium (transverse shape of string, distribution of density of fluid, etc.) which moves through the medium with a definite velocity. The velocity of propagation of the wave is not necessarily related to the velocity of any portion of the medium", [3].

"The reader may ask why we would expect an equation like (3) to describe something like the vibration of a violin string. To answer this, we must understand that the expression on the left side of the equation represents the vertical acceleration of the string at a definite point. Hence, (3) can be interpreted as saying that the acceleration of each point of the string is due to the tension in the string, and that the larger the concavity represented by the Laplacian, the stronger the force", [13].

Let us consider this in detail, because of the richness of this "Gedankenexperiment". "Let be the physical prototype a lower register piano string, which is a more or less uniformly loaded wire stretched between two fairly rigid supports. Such a string has stiffness, but experiment can show that the resistance to displacement of the string from its equilibrium shape is in the main due to the tension in the string rather than to its stiffness. Therefore, one simplification usually made in obtaining the equation governing the string's shape is that stiffness can be neglected (several books on vibration and sound analyze the effects of stiffness and show when it can safely be neglected). Other simplifying assumptions are that the mass of the string is uniformly distributed along its length, that the tension is likewise uniform, and that the displacement of a point of the string from equilibrium is always small compared with the distance of this point from the closer end support".

"The shape of such a string at any instant can be expressed in terms of its displacement from equilibrium". Now, let us picture how the piano hammer pushes up the string when the pianist presses the corresponding key. At the part of the string in direct contact with the hammer head, the displacement function of the string satisfies Poisson's equation, since it is obvious that displacement there is greater than at neighboring points. However, at the rest of the string where there is no physical contact with the hammer, the displacement function obeys Laplace's equation, since there is not any reason which could justify that displacement at any of these points was greater (or lower) than the average of its neighborhood. It is rather logical.

At the moment the pianist releases the key, the forced string is also released by the hammer, so that one may think that all parts of the string would satisfy now Laplace's equation. But what happens is that the last forced position of the string has arranged some tensional forces that tend to return the string to its original shape. Thus causing an acceleration downwards that is proportional to the upwards curvature of the string. Let us see this phenomenon reflected in the algebra of the wave equation. It is clear that an upwards curvature of the string would cause the Laplacian to be negative, and as we can see in (3), this would cause the acceleration to be also negative, that is, downwards. Then the wave equation makes sense.

Once the string begins its movement downwards to reestablish its original shape, its mass is acquiring some inertia, so that when it finally arrives at its equilibrium position, the inertia so acquired pushes it down to go on its movement, and causing this way a new displacement from the equilibrium position, but this time downwards. But a downwards displacement implies a positive Laplacian, and having a look at (3), we realize that this implies also a positive acceleration, that is upwards.

And so on, this process will repeat indefinitely causing the string to vibrate to delight the audience.

The Helmholtz Equation

Let us consider (4), which is called the Helmholtz equation. It is separated off the wave equation after some transformation, and its solution represents the spatial solution of the wave equation. From our qualitative point of view, it just means that the curvature of the displacement (represented by the Laplacian) is proportional to the actual displacement of the string (in case we go on considering this useful example). More precisely, an upwards displacement would imply a negative Laplacian, and so, a proportional positive displacement in order to satisfy (4). And viceversa.

Nevertheless, much more interesting interpretations can be made of this equation, in the light of Linear Algebra.

The Telegraphist's Equation

Equation (5) has been named so for its usefulness to study the physical behaviour of the long cables used in the dawn of telecommunications age. The details of how this equation is derived can be found in [13], and [4]. Now we are going to discuss the meaning of each term.

In principle, it is an extension of the wave equation. But we have added two negative terms, and an unknown function.

The first negative term, proportional to the velocity of the displacement is due to the effect of friction. "The effect of friction is, of course, to damp out the free vibrations". If the string is vibrating in a medium that offers a resistance to the string's velocity, then this resistance force is opposed to the movement (adding a negative term to the total acceleration), and proportional to this velocity.

The second negative term that appears in (5), and which is proportional to that actual displacement function, is meant as the restoring force, since "it is directed opposite to the displacement of the string". If the displacement is positive (upwards), then the force is negative (downwards), adding another negative term to the total acceleration.

The last term is meant to be "an external force that may be applied along the string at any position or time". It would include, for example, gravity (in which case it would be negative, since it causes always a downward acceleration term), sudden impulses along the string at different values of time, or forces that could be applied by sound waves impinging on the string.

Diffusion Equation

"In one limiting case, the viscous forces may completely predominate over the inertial effects", so that the wave equation becomes (6).

"Since it may represent the behavior of some solute diffusing through a solvent (where the function is now the density of the solute), it is usually called the diffusion equation".

"As with the wave equation, the tendency is to straighten out the curvature; however, here the velocity of any part of the string is proportional (but opposite in sign) to the curvature of that part, whereas in the wave equation it is the acceleration that is proportional (and opposite in sign) to the curvature. In short, we are essentially dealing with an equilibrium condition. In the wave equation a curved portion continually increases in velocity until it is straightened out and only then starts slowing down, thus ensuring oscillatory motion. But with the diffusion equation the velocity of any portion comes to zero when this portion is finally straightened out, so there is no oscillatory motion. One would expect this behavior for a string of no mass in a viscous fluid, since the damping would be more than critical", [5].

Equation (6) can also be used to describe the basic flow of heat, under certain conditions. The derivation of this equation from the basic conservation of heat can be found in [14], but we are going to examine it by itself. "This equation simply says that the temperature at some point is increasing or decreasing according whether its Laplacian is positive or negative", [15]. That is when the temperature at a point is higher than the average of its neighborhood, it tends to diffuse, increasing the temperature of its neighboring points at a velocity which is proportional to the original imbalance.

The proportionality constant is a property of the material.

Klein-Gordon Equation

"A type of equation of some interest in quantum mechanics can also be exemplified by the flexible string with additional stiffness forces provided by the medium surrounding the string. If the string is embedded in a thin sheet of rubber, for instance (or if it is along the axis of a cylinder of rubber whose outside surface is held fixed), then in addition to the restoring force due to tension there will be a restoring force due to the rubber on each portion of string. This restoring force will be proportional and opposite in sign to the displacement, where the proportionality constant would depend on the elastic properties and the geometrical distribution of the rubber", [6].

Therefore, the equation of motion for the string under these circumstances would be (7). This equation is called the Klein-Gordon equation when it occurs in quantum mechanical problems.

THE DIVERGENCE OPERATOR

Another very important operator in the mathematical physics is that of divergence (8). It is related with flux, and it may be understood as the net outgoing flux density for an infinitesimal closed surface. It is shown that this net outflow "depends only on the volume enclosed by the surface, that is, not on whether the volume element is for rectangular or for curvilinear coordinates". Now it is rather easy to understand the divergence theorem, since because of the additive property, the net outgoing flux integral for a "macroscopique" surface must equal the integral sum of the outgoing fluxes for all the "microscopique" elements of volume included in the region. Equation (9) is called Gauss' or divergence theorem.

"This divergence theorem emphasizes the close relationship there must be between the behavior of a vector field along a closed surface and the field's behavior everywhere in the region inside this surface". More details about this discussion can be found in [7].

Note that if the divergence is null in a region then there is not any net "microscopic" outflow, so there cannot be "divergent" lines of flow of the field "diverging" from any point source, actually there cannot be any point source. So that in a field where the divergence vanishes, the lines of flow must not converge at any point. That is precisely the case of magnetic field where lines of flow curl over themselves.

Maxwell's Equation for the Displacement Field

Equation (10) is known as Gauss' theorem in its differential form, and is one of Maxwell's equations for electrostatics, [17]. It states that the divergence of the displacement field is equal to the electrostatic density at any point.

But what is the displacement field? The electrostatic field is proportional to the displacement field, and the factor of proportionality is characteristic of the medium, as stated in (11). So that we may think of the displacement field as a more general magnitude than the electrostatic field itself.

It is rather difficult to give an explanatory definition of the electric charge. But in the light of potential distribution theory, we may say that a simple electrostatic source is a point singularity in a vector field, such that a net outflow integral of the displacement field over any surface enclosing the singularity, is equal to the strength of the source, that is the electrostatic charge. We know that it is a rather backhanded definition for a simple charge, but we just want to emphasize that the only way to detect a charge is by means of the displacement field created by its only presence. See [8] for a similar approach.

Once we agree (more or less) with previous paragraph it is easy to understand (10), since it is just a consequence of divergence and charge definitions. The divergence of the displacement field is the net outgoing flux density in the region, so that it is rather logical that it must equal the charge density, from the definition of charge stated above. Analogously, it is obvious that where there is not any charge, the divergence of the field vanishes.

Maxwell's Equation for the Magnetic Field

Equation (12) is very simple, and it just states that, as there is not known magnetic charges, the divergence of the magnetic field vanishes everywhere.

Equation of continuity

Equation (13) is called the equation of continuity for a fluid. It just means that if the divergence of the fluid is positive somewhere, that is, if it exists a net outgoing flux of mass, then there is a decrease in the mass density of the fluid at that point. The idea is very simple and rather logical.

Obviously, for a fluid of constant density (incompressible fluid), the divergence of the fluid velocity must be zero, otherwise there could not be any equilibrium of mass.

There is also an equivalent equation of continuity for electromagnetism, that is (14). It analogously means that if there is any positive divergence of the electric current, as it implies by definition that there is a net outflow of electric current, then there must be a decrease in the charge density at that point. Similarly, if charge density is fixed, then the divergence of the current density must vanish, [18].

THE CURL OPERATOR

"This operator is related to the net circulation integral, just as the divergence operator is related to the net outflow integral". We can state (without any kind of rigor) that the module of the curl of a vector function is (in a sense) a measure of its circulation density around an infinitesimal element of area, (15). No matter the shape of this elemental area. "However, it soon becomes apparent that the present limiting process is more complicated than that used to define the divergence, for the results obtained depend on the orientation of the element of area". Nevertheless, as heretics we are, we will not consider any rigorous details and refer the fussy reader to [9].

"There is a curl theorem similar to the divergence theorem expressed in previous section". From our last definition of the curl, it is obvious that to compute the net circulation integral around the boundary line of any surface, we must sum the circulations around all the elements of area included in the surface. Equation (16) is called Stoke's or curl theorem; "it enables one to compute the net circulation integral for any path".

When the curl cancels in a given domain, that implies that the field does not "curl" in the region, since there cannot be any net circulation integral. This is the case of the electric field, which instead of "curling", generally diverges from a given point.

Faraday's Law

Equation (17) states that the rate of change of the magnetic induction is proportional (but opposite in sign) to the curl of the electrostatic field. "It is called the Faraday law of electric induction and relates the change of magnetic field to the vorticity of the electric field", [10].

It basically means that if there is a closed circuit, where the electrostatic field circulates around, then obviously the curl will be positive (since it is a measure of the circulation density), and that will cause a decrease in the magnetic induction vector field, following Lenz's law. Reciprocally, a variation of the magnetic field may cause electric circulation.

Obviously, when the magnetic field is a vector constant, the curl of the electrostatic field must vanish, so that there cannot be net electric circulation.

Ampere-Maxwell Law

This last Maxwell's equation (18) states that the sum of the current density plus the rate of change of the displacement field is proportional to the curl of the magnetic field. And it means that if there is a net circulation of the magnetic field, then it will imply the curl to be nonzero, and that will cause a current density to flow, as well as an increase of the electric displacement, (which is called displacement current density, since it has units of current density). In case that the electric field is constant, then there is no variation of the displacement field. See [17] and [18].

CONCLUSION

We have dealt with some geometrical and intuitive interpretations of the main operators of the PDEs. Though these interpretations are sometimes well known, they are scarcely used in university teaching. We have tried to revive them, and put some order in them, so as to become useful pedagogical tools for our undergraduate and graduate lecturers.

ACKNOWLEDGMENT

We would like to honor Prof. Morse and Prof. Feshbach for their eminent work, so many times cited in this paper, and that we think is a richful collection of brilliant intuitive interpretations of the mathematical physics. Prof. Farlow's introduction to PDEs is also a sample of genuine creativity and originality when it deals with the explanation of the difficult mathematical concepts involved in differential operators. We would like to thank all of them for these rare contributions to university pedagogy.

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EQUATIONS

$$\nabla^2\psi = 0 \quad (1)$$

$$\nabla^2\psi = -q \quad (2)$$

$$\frac{\partial^2\psi}{\partial t^2} = c^2 \cdot \nabla^2\psi \quad (3)$$

$$\nabla^2\psi + \lambda^2 \cdot \psi = 0 \quad (4)$$

$$\frac{\partial^2\psi}{\partial t^2} = c^2 \cdot \nabla^2\psi - \beta^2 \cdot \frac{\partial\psi}{\partial t} - \gamma^2 \cdot \psi + F \quad (5)$$

$$\frac{\partial\psi}{\partial t} = \alpha^2 \cdot \nabla^2\psi \quad (6)$$

$$\frac{1}{c^2} \cdot \frac{\partial^2\psi}{\partial t^2} = \nabla^2\psi - \mu^2 \cdot \psi \quad (7)$$

$$\text{div}F = \nabla \circ F \quad (8)$$

$$\iint F \circ dS = \iiint (\nabla \circ F) \cdot dV \quad (9)$$

$$\nabla \circ D = \rho \quad (10)$$

$$D = \varepsilon \cdot E \quad (11)$$

$$\nabla \circ B = 0 \quad (12)$$

$$\frac{\partial\rho}{\partial t} = -\nabla \circ (\rho v) \quad (13)$$

$$\frac{\partial\rho}{\partial t} = -\nabla \circ J \quad (14)$$

$$\text{curl}F = \nabla \times F \quad (15)$$

$$\int F \circ dl = \iint (\nabla \times F) \circ dS \quad (16)$$

$$\frac{\partial B}{\partial t} = -\nabla \times E \quad (17)$$

$$\frac{\partial D}{\partial t} + J = \nabla \times H \quad (18)$$